CROSSED PRODUCTS BY α-SIMPLE AUTOMORPHISMS ON C*-ALGEBRAS

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Background

- A classical dynamical system consists of a compact Hausdorff space X together with a homeomorphism σ. Define $\alpha(f) = f \circ \sigma^{-1}$ for $f \in C(X)$. For any C^*-algebra A, an isomorphism from A onto itself is called an automorphism. Denote by Aut(A) the automorphism group of A. Then $\alpha \in$Aut$(C(X))$. Also $\mathbb{Z} \to$Aut(A) given by $n \to \alpha^n$ is a group homomorphism.

- Denote by $A \rtimes_\alpha \mathbb{Z}$ the crossed products by automorphisms α on C*-algebras A.

- A dynamical system (X, σ) is said to be minimal if X has no proper closed σ-invariant subset.
  If (X, σ) is minimal and X is infinite, then $C(X) \rtimes_\sigma \mathbb{Z}$ is a unital simple C^*-algebra.

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We will use the following convention:

- Let A be a C*-algebra. We denote by $\mathrm{Aut}(A)$ the automorphism group of A.
- Let A be a C*-algebra and $\alpha \in \mathrm{Aut}(A)$. We say A is α-simple if A does not have any non-trivial α-invariant closed two-sided ideals.
- Let A be a unital C*-algebra and $T(A)$ the compact convex set of tracial states of A.

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Theorem (F. Putnam)

Let X be a Cantor set and $\sigma : X \to X$ a minimal homeomorphism. Then crossed product $C(X) \rtimes_\alpha \mathbb{Z}$ is a simple AT algebras (direct limits of circle algebras) with real rank zero.

Theorem (G. Elliott and D. Evans)

Irrational rotation algebras A_θ are simple AT algebras with real rank zero.

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Theorem (G. Elliott)

Let A and B be a unital simple AT-algebra with real rank zero. If there is an order isomorphism
$\alpha : (K_0(A), K_0(A)_+, [1_A], K_1(A)) \to (K_0(B), K_0(B)_+, [1_B], K_1(B))$.
Then we have an isomorphism $h : A \to B$ such that $h_* = \alpha$.

Definition (H. Lin)

Let A be a unital simple C-algebra. Then A is said to have tracial (topological) rank zero if for any $\varepsilon > 0$, any finite set $\mathcal{F} \subset A$ and any nonzero positive element $a \in A$, there exists a finite dimensional C*-subalgebra $B \subset A$ with id$_B = p$ such that:*
(1) $\|px - xp\| < \varepsilon$ for all $x \in \mathcal{F}$.
(2) $pxp \in_\varepsilon B$ for all $x \in \mathcal{F}$.
(3) $[1-p] \leq [a]$, i.e. there is a projection $q \in \overline{aAa}$ and a partial isometry $v \in A$ such that $v^*v = 1-p$ and $vv^* = q$.

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Theorem (H. Lin)

Let A and B be unital separable simple C^*-algebras with tracial rank zero which satisfy the UCT. If there is an order isomorphism $\alpha : (K_0(A), K_0(A)_+, [1_A], K_1(A)) \to (K_0(B), K_0(B)_+, [1_B], K_1(B))$. Then there exists $h : A \to B$ such that $h_* = \alpha$.

Theorem (H. Lin and N. C. Phillips)

Let X be an infinite compact metric space with finite covering dimensional and let α: $X \to X$ be a minimal homeomorphism, the associated crossed product C-algebra $A = C(X) \rtimes_\alpha \mathbb{Z}$ has tracial rank zero whenever the image of $K_0(A)$ in Aff($T(A)$) is dense.*

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Theorem (A. S. Toms and W. Winter)

Let $\mathcal{C}$ denote the class of C^*-algebras having the following properties:
(1) every $A \in \mathcal{C}$ has the form $C(X) \rtimes_\alpha \mathbb{Z}$ for some infinite, compact, finite dimensional, metrizable space X and minimal homeomorphism $\alpha : X \to X$;
(2) the projections of every $A \in \mathcal{C}$ separate traces.
If $A, B \in \mathcal{C}$ and there is a graded ordered isomorphism $\phi : K_*(A) \to K_*(B)$, then there is a $*$-isomorphism $\Phi : A \to B$ which induces ϕ.

α-simple action on C^*-algebras

- We don't know what happen when C*-algebras are neither commutative nor simple.
- In this talk, we consider the following case: let X be a Cantor set, let A be a unital separable simple amenable C*-algebra with TR$(A \otimes M_{p^\infty}) \leq 1$ which satisfies the Universal Coefficient Theorem, we consider the C*-algebra $C(X, A)$, all continuous functions from X to A. When A is isomorphic to $\mathbb{C}$, it is just the Cantor set case. When $C(X, A)$ is not isomorphic to $\mathbb{C}$, $C(X, A)$ is neither commutative nor simple.

α-simple action on C^*-algebras

Definition

Let X be a compact metric space and let A be a C-algebra, we say a map*
$\beta : X \to$ *Aut*(A) *is strongly continuous if for any* $\{x_n\}$ *with* $d(x_n, x) \to 0$ *when* $n \to \infty$, *we have* $\|\beta_{x_n}(a) - \beta_x(a)\| \to 0$ *for all* $a \in A$.

Lemma

Let X be a compact metric space, let A be a unital simple C-algebra and $\alpha \in$Aut$(C(X, A))$. Then $C(X, A)$ is α-simple if and only if there is a minimal homeomorphism σ from X to X and there is a strongly continuous map β from X to Aut(A), denote by x to β_x, such that* $\alpha(f)(x) = \beta_{\sigma^{-1}(x)}(f(\sigma^{-1}(x)))$.

α-simple action on C^*-algebras

Lemma

Let X be an infinite compact metric space, let A be a unital simple C-algebra and $\alpha \in$Aut$(C(X, A))$. Then $C(X, A)$ is α-simple if and only if the crossed product $C(X, A) \rtimes_\alpha \mathbb{Z}$ is simple.*

Large C^*-subalgebras of crossed products

- Let u be the unitary implementing the action of α in the transformation group C*-algebra $C(X, A) \rtimes_\alpha \mathbb{Z}$, then $ufu^* = \alpha(f)$. For a nonempty closed subset $Y \subset X$, we define the C*-subalgebra B_Y to be

$$B_Y = C^*(C(X, A), uC_0(X \backslash Y, A)) \subset C(X, A) \rtimes_\alpha \mathbb{Z}.$$

We will often let B denote the transformation group C*-algebra $C(X, A) \rtimes_\alpha \mathbb{Z}$. If $Y_1 \supset Y_2 \supset \cdots$ is a decreasing sequence of closed subsets of X with $\cap_{n=1}^{\infty} Y_n = \{y\}$, then $B_{\{y\}} = \lim B_{Y_n}$.

Large C^*-subalgebras of crossed products

Lemma

Let X be an infinite compact metric space, let A be a unital simple C-algebra and $\alpha \in$Aut$(C(X, A))$. If $C(X, A)$ is α-simple, then for any $y \in X$, we have $B_{\{y\}}$ is simple.*

Lemma

Let X be a Cantor set, let A be a unital separable simple amenable C-algebra with tracial rank zero which satisfies the UCT (Universal Coefficient Theorem), and let $\alpha \in Aut(C(X, A))$. Suppose $C(X, A)$ is α-simple. It follows that for any $y \in X$, the C*-algebra $B_{\{y\}}$ has tracial rank zero.*

Large C^*-subalgebras of crossed products

Lemma

Let X be a Cantor set, let A be a stably finite unital C-algebra and $\alpha \in$Aut$(C(X,A))$. If there is a minimal homeomorphism σ from X to X and a strongly continuous map β from X to Aut(A), denote by x to β_x, such that $\alpha(f)(x) = \beta_{\sigma^{-1}(x)}(f(\sigma^{-1}(x)))$. Let $B = C(X,A) \rtimes_\alpha \mathbb{Z}$, and let $y \in X$ and Y be a clopen neighborhood of $x \in X$. Suppose for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset B$, there is a partial isometry $w \in B_{\{y\}}$ such that $w^*w = 1_Y$, $ww^* = 1_{\sigma^N(Y)}$ and $\|wf|_Y - f|_{\sigma^N(Y)}w\| < \frac{\varepsilon}{4}$ for all $f \in \mathcal{F}$. Then there is a projection $p \in B_{\{y\}}$ such that:*

(1) $\|pa - ap\| < \varepsilon$ for all $a \in \mathcal{F}$.

(2) $pap \in pB_{\{y\}}p$ for all $a \in \mathcal{F}$.

(3) $\tau(1-p) < \varepsilon$ for all $\tau \in T(B_{\{y\}})$.

Large C^*-subalgebras of crossed products

Lemma

Let X be a Cantor set, let A be a stably finite unital C-algebra and $\alpha \in$Aut$(C(X, A))$. If there is a minimal homeomorphism σ from X to X and a strongly continuous map β from X to Aut(A), denote by x to β_x, such that $\alpha(f)(x) = \beta_{\sigma^{-1}(x)}(f(\sigma^{-1}(x)))$, where $\beta_{\sigma^{-1}(x)}$ is approximately unitary equivalent to the identity map for all $x \in X$. Let $B = C(X, A) \rtimes_\alpha \mathbb{Z}$, and let $y \in X$ and Y be a clopen neighborhood of $x \in X$. Then for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset B$, there is a partial isometry $w \in B_{\{y\}}$ such that $w^*w = 1_Y$, $ww^* = 1_{\sigma^N(Y)}$ and $\|wf|_Y - f|_{\sigma^N(Y)}w\| < \frac{\varepsilon}{4}$ for all $f \in \mathcal{F}$.*

For the case that $\mathrm{TR}(A) = 0$

Theorem

Let X be a Cantor set, and let A be a unital separable simple amenable C-algebra with tracial rank zero which satisfies the UCT. Let $C(X, A)$ denote all continuous functions from X to A and α be an automorphism of $C(X, A)$. Suppose that $C(X, A)$ is α-simple and $[\alpha|_{1\otimes A}] = [id|_{1\otimes A}]$ in $KL(1 \otimes A, C(X, A))$. Then $C(X, A) \rtimes_\alpha \mathbb{Z}$ is a unital simple C*-algebra with tracial rank zero.*

For the case that TR$(A \otimes M_{p^\infty}) \leq 1$

Theorem

Let X be a Cantor set, and let A be a unital separable simple amenable C-algebra which is $\mathcal{Z}$-stable and satisfies TR$(A \otimes M_{p^\infty}) \leq 1$ and the UCT. Let $C(X, A)$ denote all continuous functions from X to A and α be an automorphism of $C(X, A)$. Suppose that*

(1) $C(X, A)$ is α-simple,

(2) $[\alpha|_{1\otimes A}] = [\mathrm{id}|_{1\otimes A}]$ in $KL(1 \otimes A, C(X, A))$,

(3) $\tau(\alpha(1 \otimes a)) = \tau(1 \otimes a)$ for all $\tau \in T(C(X, A))$, and

(4) $\alpha^{\ddagger}(\overline{1 \otimes u}) = \mathrm{id}^{\ddagger}(\overline{1 \otimes u})$ in $U_\infty(C(X, A))/CU_\infty(C(X, A))$.

Then $C(X, A) \rtimes_\alpha \mathbb{Z}$ is a unital simple C-algebra with TR$((C(X, A) \rtimes_\alpha \mathbb{Z}) \otimes M_{p^\infty}) \leq 1$.*

For the case that A is $\mathbb{T}^n$

In this section, X denotes the Cantor set, $\mathbb{T}$ denotes the circle, and $\mathbb{T}^n$ denotes the n-dimensional torus.
For a compact Hausdorff space Y, Homeo(Y) is used to denote the set of all the homeomorphisms of Y.
As the Cantor set X is totally disconnected, we can write a homeomorphism of $X \times \mathbb{T}^n$ as $\sigma \times \varphi$(the skew product form), with $\sigma \in$Homeo(X) and $\varphi : X \to$ Homeo$(\mathbb{T}^n)$ being continuous, and

$$\sigma \times \varphi : X \times \mathbb{T}^n \to X \times \mathbb{T}^n$$

defined by

$$(x, t_1, t_2, \ldots, t_n) \to (\sigma(x), \varphi(t_1, t_2, \ldots, t_n)).$$

For the case that A is $\mathbb{T}^n$

For the case that the cocycles take values in rotation groups, we can further express $\sigma \times \varphi$ as $(X \times \mathbb{T} \times \mathbb{T} \times \cdots \times \mathbb{T}, \sigma \times R_{\xi_1} \times R_{\xi_2} \cdots \times R_{\xi_n})$, with $R_{\xi_n} : X \to \mathbb{T}$ continuous, and

$$\sigma \times R_{\xi_1} \times R_{\xi_2} \cdots \times R_{\xi_n} : X \times \mathbb{T}^n \to X \times \mathbb{T}^n$$

defined by

$$(x, t_1, t_2, \ldots, t_n) \to (\sigma(x), t_1 + \xi_1(x), t_2 + \xi_2(x), \ldots, t_n + \xi_n(x)).$$

For the case that A is $\mathbb{T}^n$

Lemma

If $(\sigma \times R_{\xi_1} \times R_{\xi_2} \cdots \times R_{\xi_n}, X \times \mathbb{T}^n)$ *is minimal, then for any* $y \in X$, $B_{\{y\}}$ *has tracial rank one.*

For the case that A is $\mathbb{T}^n$

Theorem

Let $X \times \mathbb{T}_1 \times \cdots \times \mathbb{T}_n$ to denote the product of the Cantor set and n dimensional torus. Let $\sigma \times R_{\xi_1} \times \cdots \times R_{\xi_n}$ be a minimal homeomorphism on $X \times \mathbb{T}_1 \times \cdots \times \mathbb{T}_n$. Let B be the crossed product C^*-algebra $C^*(\mathbb{Z}, X \times \mathbb{T}_1 \times \cdots \times \mathbb{T}_n, \sigma \times R_{\xi_1} \times \cdots \times R_{\xi_n})$. Then TR$(B) \leq 1$.

Thank you!